Twelfth International Olympiad, 1970

1970/1.

Let M be a point on the side AB of ΔABC. Let r_1, r_2 and r be the radii of the inscribed circles of triangles AMC, BMC and ABC. Let q_1, q_2 and q be the radii of the escribed circles of the same triangles that lie in the angle ACB. Prove that

$$\frac{r_1}{q_1} \cdot \frac{r_2}{q_2} = \frac{r}{q}.$$

1970/2.

Let a, b and n be integers greater than 1, and let a and b be the bases of two number systems. A_{n-1} and A_n are numbers in the system with base a, and B_{n-1} and $B_n$are numbers in the system with base b; these are related as follows:

$$\begin{aligned}
A_n &= x_n x_{n-1} \cdots x_0, A_{n-1} = x_{n-1} x_{n-2} \cdots x_0, \\
B_n &= x_n x_{n-1} \cdots x_0, B_{n-1} = x_{n-1} x_{n-2} \cdots x_0, \\
x_n &\neq 0, x_{n-1} \neq 0.
\end{aligned}$$

Prove:

$$\frac{A_{n-1}}{A_n} < \frac{B_{n-1}}{B_n} \text{ if and only if } a > b.$$

1970/3.

The real numbers $a_0, a_1, ..., a_n, ...$ satisfy the condition:

$$1 = a_0 \leq a_1 \leq a_2 \leq \cdots \leq a_n \leq \cdots.$$

The numbers $b_1, b_2, ..., b_n, ...$ are defined by

$$b_n = \sum_{k=1}^{n} \left(1 - \frac{a_{k-1}}{a_k}\right) \frac{1}{\sqrt{a_k}}.$$

(a) Prove that $0 \leq b_n < 2$ for all n.
(b) Given c with $0 \leq c < 2$, prove that there exist numbers $a_0, a_1, ...$ with the above properties such that $b_n > c$ for large enough n.

1970/4.

Find the set of all positive integers n with the property that the set $\{n, n+1, n+2, n+3, n+4, n+5\}$ can be partitioned into two sets such that the product of the numbers in one set equals the product of the numbers in the other set.

1970/5.

In the tetrahedron $ABCD$, angle BDC is a right angle. Suppose that the foot H of the perpendicular from D to the plane ABC is the intersection of the altitudes of ΔABC. Prove that

$$(AB + BC + CA)^2 \leq 6(AD^2 + BD^2 + CD^2).$$

For what tetrahedra does equality hold?

1970/6.

In a plane there are 100 points, no three of which are collinear. Consider all possible triangles having these points as vertices. Prove that no more than 70% of these triangles are acute-angled.